The Correspondence Principle

The Correspondence Principle is a belief in physics that any new laws discovered must reduce to the classic laws in the appropriate conditions. (It was first proposed by Neils Bohr as he introduced the idea of quantum energy levels to develop a model of the atom that could explain a variety of experimental results.) This means that special relativity must be used in speeds approaching that of light, but at low speed, the equations of special relativity must reduce to the classical equations of motion.

It is easy to see how this applies to momentum: at low speeds $\gamma \approx 1$, so therefore

$$p = \gamma mv \approx mv$$

It is not so easy to see how this applies to kinetic energy, as by the same logic we would say:

$$K = (\gamma - 1)E_0 \approx (1-1)E_0 = 0!$$

The problem is that the rest energy of an object is HUGE so the kinetic energy it has because it is moving at 5 m/s is indescribably tiny compared to the rest energy. We need to show that the expression we just derived for kinetic energy must be approximately the same for the classical definition, K = 1/2 mv^2. To do this, we will use the binomial theorem.

$$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \ldots$$

Rewriting the kinetic energy we just derived, and then applying the first two terms of the binomial theorem where $x = -v^2/c^2$ and $n = -1/2$:

$$\begin{aligned} K &= mc^2\left(\left(1 - {v^2}/{c^2}\right)^{-1/2} - 1\right) \\ &\approx mc^2\left[\left(1 + \frac{\left(-{1}/{2}\right)\left(-{v^2}/{c^2}\right)}{1!}\right) - 1\right] \\ &\approx mc^2\left(1 + {v^2}/{2c^2} - 1\right) \\ K &\approx {1}/{2}\, mv^2 \end{aligned}$$

So $1/2mv^2$ is just an approximation for the actual kinetic energy. While dropping all those terms in the binomial expansion seems kind of ridiculous, the maximum amount of error when dropping those terms is actually the value of the first term dropped, so the maximum amount of error is

$$\begin{aligned} error &\approx \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2} - 1\right)\left(-{v^2}/{c^2}\right)^2}{2!} \\ &\approx {v^4}/{c^4} \end{aligned}$$

For everyday objects, v/c is very small, so $(v/c)^4$ is extremely small. Even the orbital speed of the earth, about 30 km/s, is only 10^{-4} the speed of light, so that the error is going to be 10^{-16} times smaller than the actual kinetic energy.